

CLASSIFICATION OF HOMOTHETICAL HYPERSURFACES AND ITS APPLICATIONS TO PRODUCTION FUNCTIONS IN ECONOMICS

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Abstract. In this paper, we completely classify the homothetical hypersurfaces having null Gauss-Kronocker curvature in a Euclidean $(n + 1)$ –space \mathbb{R}^{n+1} . Several applications to the production functions in economics are also given.

Key words: Homothetical hypersurface, Gauss-Kronocker curvature, production function, Cobb-Douglas production function, ACMS production function, Elasticity of substitution.

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1. Introduction and preliminaries on hypersurfaces

A hypersurface M^n of a Euclidean $(n + 1)$ –space \mathbb{R}^{n+1} is called a *homothetical hypersurface* if it is the graph of a function of the form:

$$f(x_1, \dots, x_n) = f_1(x_1) \times \dots \times f_n(x_n), \quad (1.1)$$

where f_1, \dots, f_n are functions of class C^∞ [25]. We call f_1, \dots, f_n the *components* of f , and denote the homothetical hypersurface M^n by a pair (M^n, f) .

Homothetical hypersurfaces have been studied by many authors based on minimality property of these hypersurfaces [18, 20, 24, 25, 27].

G.E. Vilcu and A.D. Vilcu [28, 29] established an interesting link between some fundamental notions in the theory of production functions and the differential geometry of hypersurfaces. For further study of production hypersurfaces, we refer the reader to B.-Y. Chen's series of interesting papers on homogeneous production functions, quasi-sum production models, and homothetic production functions [5, 6, 8-15] and X. Wang and Y. Fu's paper [30].

Let M^n be a hypersurface of a Euclidean $(n + 1)$ –space \mathbb{R}^{n+1} . For general references on the geometry of hypersurfaces see [7, 17, 19].

The *Gauss map* $\nu : M^n \longrightarrow S^{n+1}$ maps M^n to the unit hypersphere S^n of \mathbb{R}^{n+1} . The differential $d\nu$ of the Gauss map ν is known as *shape operator* or Weingarten map. Denote by $T_p M^n$ the tangent space of M^n at the point $p \in M^n$. Then, for $v, w \in T_p M^n$, the shape operator A_p at the point $p \in M^n$ is defined by

$$g(A_p(v), w) = g(d\nu(v), w),$$

where g is the induced metric tensor on M^n from the Euclidean metric on \mathbb{R}^{n+1} .

The determinant of the shape operator A_p is called the *Gauss-Kronocker curvature*. A hypersurface having null Gauss-Kronecker curvature is said to be *developable*. In this case the hypersurface can be flattened onto a hyperplane without distortion. We remark that cylinders and cones are examples of developable surfaces, but the spheres are not under any metric.

For a given function $f = f(x_1, \dots, x_n)$, the graph of f is the non-parametric hypersurface of \mathbb{R}^{n+1} defined by

$$\varphi(\mathbf{x}) = (x_1, \dots, x_n, f(\mathbf{x})) \quad (1.2)$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Let us put

$$\omega = \sqrt{1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \right)^2}. \quad (1.3)$$

The Gauss-Kronecker curvature of the graph of f is

$$G = \frac{\det(\mathcal{H}(f))}{\omega^{n+2}}, \quad (1.4)$$

where $\mathcal{H}(f)$ is the Hessian matrix of f , that is, the square matrix $(f_{x_i x_j})$ of second-order partial derivatives of f .

In this paper, we completely classify homothetical hypersurfaces having null Gauss-Kronecker curvature. Several applications to production models in economics are also given.

2. Production models in economics

In economics, a *production function* is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+, \quad f = f(x_1, x_2, \dots, x_n),$$

where f is the quantity of output, n are the number of inputs and x_1, x_2, \dots, x_n are the inputs.

A production function $f(x_1, x_2, \dots, x_n)$ is said to be *homogeneous of degree p* or *p -homogenous* if

$$f(tx_1, \dots, tx_n) = t^p f(x_1, \dots, x_n) \quad (2.1)$$

holds for each $t \in \mathbb{R}_+$ for which (2.1) is defined. A homogeneous function of degree one is called *linearly homogeneous*. If $h > 1$, the function exhibits increasing return to scale, and it exhibits decreasing return to scale if $h < 1$. If it is homogeneous of degree 1, it exhibits constant return to scale [8].

In 1928, C. W. Cobb and P. H. Douglas introduces [16] a famous two-factor production function

$$Y = bL^k C^{1-k},$$

where b presents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called *Cobb-Douglas production function*. In its generalized form the Cobb-Douglas production function may be expressed as

$$f(\mathbf{x}) = \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n},$$

where γ is a positive constant and $\alpha_1, \dots, \alpha_n$ are nonzero constants.

In 1961, K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [2] introduced a two-factor production function given by

$$Q = F \cdot (aK^r + (1 - a)L^r)^{\frac{1}{r}},$$

where Q is the output, F the factor productivity, a the share parameter, K and L the primary production factors, $r = (s - 1)/s$, and $s = 1/(1 - r)$ is the elasticity of substitution.

The *generalized ACMS production function of n variables* is given by

$$f(x_1, \dots, x_n) = \gamma (\beta_1^p x_1^p + \dots + \beta_n^p x_n^p)^{\frac{d}{p}},$$

where $\rho \neq 0$, $\rho < 1$, $d, \gamma > 0$ and $\beta_i > 0$ for all $i = 1, \dots, n$.

A *homothetic function* is a production function of the form:

$$f(x_1, \dots, x_n) = F(h(x_1, \dots, x_n)),$$

where $h(x_1, \dots, x_n)$ is homogeneous function of arbitrary given degree and F is a monotonically increasing function [12, 15, 21].

A homothetic production function of form

$$f(\mathbf{x}) = F\left(\sum_{i=1}^n \beta_i^\rho x_i^\rho\right)^{\frac{d}{\rho}} \quad (\text{resp., } f(\mathbf{x}) = F(x_1^{\alpha_1} \dots x_n^{\alpha_n}))$$

is called a *homothetic generalized ACMS production function* (resp., a *homothetic generalized Cobb-Douglas production function*) [11].

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. R.G.D. Allen and J.R. Hicks [1] suggested two generalizations of Hicks' original two variable elasticity concept.

The first concept, called Hicks elasticity of substitution, is defined as follows.

Let $f(x_1, \dots, x_n)$ be a production function. Then *Hicks elasticity of substitution* of the i -th production variable with respect to the j -th production variable is given by

$$H_{ij}(\mathbf{x}) = -\frac{\frac{1}{x_i f_{x_i}} + \frac{1}{x_j f_{x_j}}}{\frac{f_{x_i x_i}}{(f_{x_i})^2} - \frac{2f_{x_i x_j}}{f_{x_i} f_{x_j}} + \frac{f_{x_j x_j}}{(f_{x_j})^2}} \quad (\mathbf{x} \in \mathbb{R}_+^n, \quad i, j = 1, \dots, n, \quad i \neq j), \quad (2.2)$$

where $f_{x_i} = \partial f / \partial x_i$, $f_{x_i x_j} = \partial^2 f / \partial x_i \partial x_j$.

A production function f is said to satisfy the constant Hicks elasticity of substitution property if there is a nonzero constant $\sigma \in \mathbb{R}$ such that

$$H_{ij}(\mathbf{x}) = \sigma, \quad \text{for } \mathbf{x} \in \mathbb{R}_+^n \text{ and } 1 \leq i \neq j \leq n. \quad (2.3)$$

L. Losonczi [22] classified homogeneous production functions of 2 variables, having constant Hicks elasticity of substitution. Then, the classification of L. Losonczi was extended to n variables by B-Y. Chen [13].

The second concept, investigated by R.G.D. Allen and H. Uzawa [26], is the following:

Let f be a production function. Then *Allen elasticity of substitution* of the i -th production variable with respect to the j -th production variable is defined by

$$A_{ij}(\mathbf{x}) = -\frac{x_1 f_{x_1} + x_2 f_{x_2} + \dots + x_n f_{x_n}}{x_i x_j} \frac{D_{ij}}{\det(\mathcal{H}^B(f))} \quad (\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n, \quad i, j = 1, \dots, n, \quad i \neq j), \quad (2.4)$$

where D is the determinant of the bordered Hessian matrix

$$\mathcal{H}^B(f) = \begin{pmatrix} 0 & f_{x_1} & \dots & f_{x_n} \\ f_{x_1} & f_{x_1 x_1} & \dots & f_{x_1 x_n} \\ \vdots & \vdots & \dots & \vdots \\ f_{x_n} & f_{x_n x_1} & \dots & f_{x_n x_n} \end{pmatrix}, \quad (2.5)$$

and D_{ij} is the co-factor of the element $f_{x_i x_j}$ in the determinant $\det(\mathcal{H}^B(f))$ ($\det(\mathcal{H}^B(f)) \neq 0$ is assumed). The authors call the bordered Hessian matrix $\mathcal{H}^B(f)$ by *Allen's matrix* and $\det \mathcal{H}^B(f)$ by *Allen determinant* in [3, 4].

It is a simple calculation to show that in case of two variables Hicks elasticity of substitution coincides with Allen elasticity of substitution.

3. Classification of homothetical hypersurfaces

Throughout this article, we assume that the functions $f_1(x_1), \dots, f_n(x_n)$ are real valued functions and have non-vanishing first derivatives.

The following provides an useful formula for Hessian determinant of a function of the form (1.1)

Lemma 3.1. *The determinant of the Hessian matrix of the function $f(\mathbf{x}) = f_1(x_1) \times \dots \times f_n(x_n)$ is given by*

$$\det(H(f)) = (f)^n \left[\frac{f_1''}{f_1} \prod_{i=2}^n \left(\frac{f_i'}{f_i} \right)' + \left(\frac{f_1'}{f_1} \right)' \sum_{i=2}^n \left(\frac{f_2'}{f_2} \right)' \dots \left(\frac{f_{i-1}'}{f_{i-1}} \right)' \left(\frac{f_i'}{f_i} \right)^2 \left(\frac{f_{i+1}'}{f_{i+1}} \right)' \dots \left(\frac{f_n'}{f_n} \right)' \right],$$

where $f_i' = \frac{df}{dx_i}$, $f_i'' = \frac{d^2 f}{dx_i^2}$ for all $i \in \{1, \dots, n\}$.

Proof. Let f be a twice differentiable function given by

$$f(\mathbf{x}) = f_1(x_1) \times \dots \times f_n(x_n) \quad (3.1)$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. It follows from (3.1) that

$$f_{x_i} = \frac{f_i'}{f_i} f, \quad f_{x_i x_j} = \frac{f_i' f_j'}{f_i f_j} f, \quad f_{x_i x_i} = \frac{f_i''}{f_i} f, \quad 1 \leq i \neq j \leq n. \quad (3.2)$$

By using (3.2), the Hessian determinant of the function f is

$$\det(H(f)) = (f)^n \begin{vmatrix} \frac{f_1''}{f_1} & \frac{f_1' f_2'}{f_1 f_2} & \frac{f_1' f_3'}{f_1 f_3} & \dots & \frac{f_1' f_n'}{f_1 f_n} \\ \frac{f_1' f_2'}{f_1 f_2} & \frac{f_1' f_2''}{f_1 f_2} & \frac{f_1' f_3'}{f_1 f_3} & \dots & \frac{f_1' f_n'}{f_1 f_n} \\ \frac{f_1' f_3'}{f_1 f_3} & \frac{f_2' f_3'}{f_2 f_3} & \frac{f_2' f_3''}{f_2 f_3} & \dots & \frac{f_2' f_n'}{f_2 f_n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \frac{f_1' f_n'}{f_1 f_n} & \frac{f_2' f_n'}{f_2 f_n} & \frac{f_3' f_n'}{f_3 f_n} & \dots & \frac{f_n''}{f_n} \end{vmatrix}. \quad (3.3)$$

Now we apply Gauss elimination method for the determinant from the formula (3.3). We replace the second column by second column minus $\left(\frac{f_1 f_2'}{f_1' f_2}\right)$ times first column; then we derive

$$\det(H(f)) = (f)^n \begin{vmatrix} \frac{f_1''}{f_1} & -\frac{f_1 f_2'}{f_1' f_2} \left(\frac{f_1'}{f_1}\right)' & \frac{f_1' f_3'}{f_1 f_3} & \cdots & \frac{f_1' f_n'}{f_1 f_n} \\ \frac{f_1' f_2'}{f_1 f_2} & \left(\frac{f_2'}{f_2}\right)' & \frac{f_2' f_3'}{f_2 f_3} & \cdots & \frac{f_2' f_n'}{f_2 f_n} \\ \frac{f_1' f_3'}{f_1 f_3} & 0 & \frac{f_3''}{f_3} & \cdots & \frac{f_3' f_n'}{f_3 f_n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{f_1' f_n'}{f_1 f_n} & 0 & \frac{f_3' f_n'}{f_3 f_n} & \cdots & \frac{f_n''}{f_n} \end{vmatrix}$$

By similar elementary transformations, we get

$$\det(H(f)) = (f)^n \begin{vmatrix} \frac{f_1''}{f_1} & -\frac{f_1 f_2'}{f_1' f_2} \left(\frac{f_1'}{f_1}\right)' & -\frac{f_1 f_3'}{f_1' f_3} \left(\frac{f_1'}{f_1}\right)' & \cdots & -\frac{f_1 f_n'}{f_1' f_n} \left(\frac{f_1'}{f_1}\right)' \\ \frac{f_1' f_2'}{f_1 f_2} & \left(\frac{f_2'}{f_2}\right)' & 0 & \cdots & 0 \\ \frac{f_1' f_3'}{f_1 f_3} & 0 & \left(\frac{f_3'}{f_3}\right)' & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{f_1' f_n'}{f_1 f_n} & 0 & 0 & \cdots & \frac{f_n''}{f_n} \end{vmatrix}. \quad (3.4)$$

After calculating the determinant from the formula (3.4), we finally obtain

$$\det(H(f)) = (f)^n \left[\frac{f_1''}{f_1} \left(\frac{f_2'}{f_2}\right)' \cdots \left(\frac{f_n'}{f_n}\right)' + \left(\frac{f_1'}{f_1}\right)' \sum_{i=2}^n \left(\frac{f_2'}{f_2}\right)' \cdots \left(\frac{f_{i-1}'}{f_{i-1}}\right)' \left(\frac{f_i'}{f_i}\right)^2 \left(\frac{f_{i+1}'}{f_{i+1}}\right)' \cdots \left(\frac{f_n'}{f_n}\right)' \right].$$

Next result completely classifies the homothetical hypersurfaces having null Gauss-Kronocker curvature.

Theorem 3.1. *Let (M^n, f) be a homothetical hypersurface in \mathbb{R}^{n+1} . (M^n, f) has null Gauss-Kronocker curvature if and only if it is parametrized by one of the following*

- (a) $\varphi(\mathbf{x}) = (x_1, \dots, x_n, f_1(x_1) \times \gamma e^{\lambda_2 x_2 + \lambda_3 x_3} \times \dots \times f_n(x_n))$ for nonzero constants $\gamma, \lambda_2, \lambda_3$;
- (b) $\varphi(\mathbf{x}) = (x_1, \dots, x_n, \gamma(x_1 + \beta_1)^{\alpha_1} \times \dots \times (x_n + \beta_n)^{\alpha_n})$, where β_1, \dots, β_n are some constants $\gamma, \alpha_1, \dots, \alpha_n$ nonzero constants such that $\sum_{i=1}^n \alpha_i = 1$.

Proof. Let (M^n, f) be a homothetical hypersurface in \mathbb{R}^{n+1} parametrized by

$$\varphi(\mathbf{x}) = (x_1, \dots, x_n, f_1(x_1) \times \dots \times f_n(x_n)).$$

Assume that (M^n, f) has null Gauss-Kronocker curvature. It follows from (1.4) that $\det(H(f)) = 0$. Hence by Lemma 3.1, we get

$$\frac{f_1''}{f_1} \prod_{i=2}^n \left(\frac{f_i'}{f_i}\right)' + \left(\frac{f_1'}{f_1}\right)' \sum_{i=2}^n \left(\frac{f_2'}{f_2}\right)' \cdots \left(\frac{f_{i-1}'}{f_{i-1}}\right)' \left(\frac{f_i'}{f_i}\right)^2 \left(\frac{f_{i+1}'}{f_{i+1}}\right)' \cdots \left(\frac{f_n'}{f_n}\right)' = 0. \quad (3.5)$$

For the equation (3.5) we have two cases:

Case (i): At least one of $\left(\frac{f'_1}{f_1}\right)', \dots, \left(\frac{f'_n}{f_n}\right)'$ vanishes. Without loss of generality, we may assume that

$$\left(\frac{f'_2}{f_2}\right)' = 0. \quad (3.6)$$

Thus from (3.5) we have

$$\left(\frac{f'_1}{f_1}\right)' \left(\frac{f'_2}{f_2}\right)^2 \left(\frac{f'_3}{f_3}\right)' \dots \left(\frac{f'_n}{f_n}\right)' = 0. \quad (3.7)$$

Without loss of generality, we may assume from (3.7) that

$$\left(\frac{f'_3}{f_3}\right)' = 0. \quad (3.8)$$

By solving (3.6) and (3.8), we conclude the following

$$f_2(x_2) = \gamma_2 e^{\lambda_2 x_2}, \quad f_3(x_3) = \gamma_3 e^{\lambda_3 x_3}$$

for nonzero constants $\gamma_2, \gamma_3, \lambda_2, \lambda_3$. This gives the statement (a) of the theorem.

Case (ii): $\left(\frac{f'_1}{f_1}\right)', \dots, \left(\frac{f'_n}{f_n}\right)'$ are nonzero. Then from (3.5), by dividing with the product $\left(\frac{f'_1}{f_1}\right)' \times \dots \times \left(\frac{f'_n}{f_n}\right)'$, we write

$$\frac{\frac{f''_1}{f_1}}{\left(\frac{f'_1}{f_1}\right)'} + \left\{ \frac{\left(\frac{f'_2}{f_2}\right)^2}{\left(\frac{f'_2}{f_2}\right)'} + \dots + \frac{\left(\frac{f'_n}{f_n}\right)^2}{\left(\frac{f'_n}{f_n}\right)'} \right\} = 0. \quad (3.9)$$

We divide the proof of case (ii) into two cases.

Case (ii.a): Taking the partial derivative of (3.9) with respect to x_i for $i = 2, 3, \dots, n$, we have

$$\left(\frac{f'_i}{f_i}\right) \left(\frac{f'_2}{f_2}\right)'' = 2 \left[\left(\frac{f'_i}{f_i}\right)' \right]^2. \quad (3.10)$$

By solving (3.10), we find

$$f_i(x_i) = \gamma_i (x_i + \beta_i)^{\alpha_i}, \quad 2 \leq i \leq n \quad (3.11)$$

for some nonzero constants α_i, γ_i and some constants β_i .

Case (ii.b): Taking the partial derivative of (3.9) with respect to x_1 , we get

$$\frac{f_1'''(x_1)}{f_1''(x_1)} + \frac{f_1'(x_1)}{f_1(x_1)} = 2 \frac{f_1''(x_1)}{f_1'(x_1)},$$

which implies that

$$f_1(x_1) f_1''(x_1) = \tau (f_1'(x_1))^2 \quad (3.12)$$

for some nonzero constant τ .

Now, we divide the proof of case(ii.b) into two cases based on the value of τ .

Case (ii.b.1): $\tau = 1$. This case is not possible because of $\left(\frac{f'_1}{f_1}\right)', \dots, \left(\frac{f'_n}{f_n}\right)'$ are nonzero.

Case (ii.b.2): $\tau \neq 1$. After solving (3.12), we derive

$$f_1(x_1) = \gamma_1(x_1 + \beta_1)^{-\frac{1}{\tau-1}}. \quad (3.13)$$

By substituting (3.11) and (3.13) into (3.9), we deduce that $\alpha_2 + \dots + \alpha_n = \tau/\tau - 1$. Therefore we obtain case (b) of the theorem.

Conversely, it is direct to verify all homothetical hypersurfaces parametrized by cases (a) and (b) have null Gauss-Kronocker curvature.

4. Applications to Cobb-Douglas production functions

Geometric representation of the generalized Cobb-Douglas production is given by the hypersurface $\varphi : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^{n+1}$,

$$\varphi(\mathbf{x}) = (x_1, \dots, x_n, \gamma x_1^{\alpha_1} \dots x_n^{\alpha_n}),$$

which is called the *Cobb-Douglas hypersurface* [28]. G. E. Vilcu [28] proved that a generalized Cobb-Douglas hypersurface is developable if and only if it has constant return to scale, i.e., $\sum_{i=1}^n \alpha_i = 1$. Thus we have the following as a consequence of Theorem 4.1:

Corollary 4.1. *Let (M^n, f) be a homothetical hypersurface in \mathbb{R}_+^{n+1} such that all components of f satisfy $\left(\frac{f'_i}{f_i}\right)' \neq 0$. (M^n, f) has null Gauss-Kronocker curvature if and only if, up to constants, it is a generalized Cobb-Douglas hypersurface having constant return to scale.*

On the other hand, assume that $h_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ ($i = 1, \dots, n$) and $F : I \subset \mathbb{R} \longrightarrow \mathbb{R}_+$ are non-vanishing differentiable functions having nonzero first derivatives. Then for $h_1(x_1) \times \dots \times h_n(x_n) \in I$, we have the following composite function

$$f(x_1, \dots, x_n) = F(h_1(x_1) \times \dots \times h_n(x_n)). \quad (4.1)$$

The authors obtained following result in [4]:

Theorem 4.1. [4] *Let $F(u)$ be a twice differentiable function with $F'(u) \neq 0$ and let f be a composite function given by*

$$f = F(h_1(x_1) \times \dots \times h_n(x_n)),$$

where h_1, \dots, h_n are thrice differentiable and nonzero functions. Then the Allen matrix $M(f)$ of f is singular if and only if f is one of the following:

- (a) $f = F(\gamma e^{\alpha_1 x_1 + \alpha_2 x_2} \times h_3(x_3) \times \dots \times h_n(x_n))$, where $\gamma, \alpha_1, \alpha_2$ are nonzero constants;
- (b) $f = F(\gamma (x_1 + \beta_1)^{\alpha_1} \times \dots \times (x_n + \beta_n)^{\alpha_n})$, where γ, α_i are nonzero constants satisfying $\alpha_1 + \dots + \alpha_n = 0$ and β_i some constants.

Thus, from Theorem 3.1 and Theorem 4.1, we have the following

Corollary 4.2. *Let (M^n, f) be a homothetical hypersurface in \mathbb{R}_+^{n+1} such that at least one of $\left(\frac{f'_1}{f_1}\right)', \dots, \left(\frac{f'_n}{f_n}\right)'$ vanishes. (M^n, f) has null Gauss-Kronocker curvature if and only if the Allen matrix $H^B(f)$ of f is singular.*

5. A Further Application

Next result completely classifies the composite functions of the form (4.1) having constant Hicks elasticity of substitution property.

Theorem 5.1. *Let $f(\mathbf{x}) = F(h_1(x_1) \times \dots \times h_n(x_n))$ be a twice differentiable production function. Then f satisfies constant Hicks elasticity of substitution property if and only if, up to constants, f is one of the following*

(a) *a homothetical generalized Cobb-Douglas production function given by*

$$f = F(x^{\alpha_1} \dots x^{\alpha_n});$$

(b) *a homothetical generalized ACMS production function given by*

$$f = F\left(\beta_1 x_1^{\frac{\sigma-1}{\sigma}} + \dots + \beta_n x_n^{\frac{\sigma-1}{\sigma}}\right), \quad \sigma \neq 1;$$

(c) $f = F\left(\prod_{i=1}^n \ln(x_i)^{\mu_i}\right)$, where μ_i are nonzero constants for all $i \in \{1, \dots, n\}$.

Proof. Let f be a twice differentiable production function given by

$$f(\mathbf{x}) = F(h_1(x_1) \times \dots \times h_n(x_n)). \quad (5.1)$$

It follows from (5.1) that

$$f_{x_i} = \frac{h'_i}{h_i} u F', \quad f_{x_i x_i} = \frac{h''_i}{h_i} u F' + \left(\frac{h'_i}{h_i}\right)^2 u^2 F'' \quad (5.2)$$

and

$$f_{x_i x_j} = \frac{h'_i h'_j}{h_i h_j} u (F' + u F''), \quad 1 \leq i \neq j \leq n, \quad (5.3)$$

where $u = h_1(x_1) \times \dots \times h_n(x_n)$. By substituting (5.2) and (5.3) into (2.2), we deduce that

$$\frac{h''_i h_i}{(h'_i)^2} + \frac{h_i}{\sigma x_i h'_i} + \frac{h''_j h_j}{(h'_j)^2} + \frac{h_j}{\sigma x_j h'_j} = 2. \quad (5.4)$$

From (5.4), we have

$$\frac{h''_i h_i}{(h'_i)^2} + \frac{h_i}{\sigma x_i h'_i} = \zeta_i, \quad (5.5)$$

where ζ_i are nonzero constants such that $\zeta_i + \zeta_j = 2$ for $1 \leq i \neq j \leq n$.

Now we divide the proof into two separate cases.

Case (i): $\zeta_i = 1 = \zeta_j$ for $1 \leq i \neq j \leq n$. After solving (5.5), we find

$$h_i(x_i) = \begin{cases} \gamma_i x_i^{\alpha_i} & \text{if } \sigma = 1 \\ e^{\kappa_i + \gamma_i \left(\frac{\sigma-1}{\sigma}\right) x_i^{\frac{\sigma-1}{\sigma}}} & \text{if } \sigma \neq 1 \end{cases} \quad (5.6)$$

for nonzero constants γ_i, α_i and some constant κ_i . Combining (5.1) and (5.6) gives cases (a) and (b) of the theorem.

Case (ii): $\zeta_i \neq 1 \neq \zeta_j$ for $1 \leq i \neq j \leq n$. By solving (5.5), we derive

$$h_i(x_i) = \begin{cases} \left((1 - \zeta_i) \kappa_i + \ln x_i^{(1-\zeta_i)\eta_i} \right)^{\frac{1}{1-\zeta_i}} & \text{if } \sigma = 1 \\ \left(\left(\frac{(1-\zeta_i)\eta_i\sigma}{\sigma-1} \right) x_i^{\frac{\sigma-1}{\sigma}} + (1 - \zeta_i) \kappa_i \right)^{\frac{1}{1-\zeta_i}} & \text{if } \sigma \neq 1 \end{cases} \quad (5.7)$$

for nonzero constants η_i and some constants κ_i . Now, by combining (5.1) and (5.7), up to constants, we obtain the cases (a) and (c) of the theorem.

Conversely, it is straightforward to verify that each one of cases (a)-(c) implies that f satisfies constant Hicks elasticity of substitution property.

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